Inverted Dagum Distribution: Statistical Properties and Simulation

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Abstract

Classical distributions do not always provide a reasonable fit for all types of datasets; therefore, there is a need to generalize existing distributions to enhance their flexibility in modeling data. This article introduces a novel univariate probability distribution termed the inverted Dagum distribution. An extensive analysis of the statistical properties of this distribution was conducted, including the hazard function, survival function, Renyi's entropy, and quantile function of the order statistics. The PDF and hazard function plots of the new distribution were displayed, indicating a right-skewed PDF with increasing, decreasing, and unimodal hazard rate shapes. Parameter estimation of the model was performed using the maximum likelihood method, and the consistency of the estimates was validated through Monte Carlo simulation. The results show that the MLE is reliable and consistent.

Keywords: Dagum distribution, Inverse transformation, Maximum likelihood estimation and Monte Carlo simulation.

1. Introduction

Dagum distribution was named after Camilo Dagum who proposed it in 1970s to fit wealth and income data as well as accommodating heavy tailed models. Dagum distribution can be of two forms (the three parameter and the four parameter) respectively refers to as Dagum type I distribution (Dagum, 1977) and Dagum type II distribution (Dagum, 1980).

Definition: A random variable $\boldsymbol{X}$ is said to have type I Dagum distribution with parameter α, λ, θ if its cumulative density function and probability density function are given by:

$$\boldsymbol{F}_{\alpha_{DD}}(\boldsymbol{x};\theta,\alpha,\lambda)=\left[1+\lambda\boldsymbol{x}^{-\alpha}\right]^{-\theta} \qquad \boldsymbol{x}>0 \tag{1}$$

$$\boldsymbol{f}_{\alpha_{DD}}(\boldsymbol{x};\theta,\alpha,\lambda)=\theta\alpha\lambda\boldsymbol{x}^{-\alpha-1}\left(1+\lambda\boldsymbol{x}^{-\alpha}\right)^{-\theta-1} \tag{2}$$

Where $\alpha,\theta,\lambda>0$. With λ as the scale parameter and α,θ as the two shape parameters. The later control the tail weight and the former control the size of the distribution. It can be observed that for $\theta=1$, (1) becomes the log logistic distribution proposed by (McDonald, 1984). It can also be observed that (1) is a Burr III distribution with an additional λ. One crucial characteristic possessed by Dagum distribution is that, in addition to its flexibility, its hazard function can be decreased, up-side down, bathtub, and then up-side bathtub shaped (see Domma, 2002). A lot of researchers utilized this behavior to study the Dagum distribution in several fields. An extensive review of the Dagum distribution and its application was detailed in (Kleiber & Kotz, 2003) and (Kleiber, 2008). The parameters of the Dagum distribution with censored samples was studied by (Domma, et al., 2011) while (Shahzad, 2013) utilized TL-moments for similar purpose. Some classs of weighted Dagum and related distributions were proposed by (Oluyede & Ye, 2014) and the five-parameter beta-Dagum by (Domma & Condino, 2013). Considering the properties of McDonald, Kumaraswamy and Dagum distribution, two new hybrid distributions called Mc-Dagum and Kum-Dagum distributions were proposed by (Oluyede & Rajasooriya, 2013). Numerous ways of extending well-known distributions were suggested by applied statisticians. An extended Dagum, distribution was proposed by (Gomes-Silva et al., 2017). The distribution of the reciprocal of any well-known distribution are their corresponding inverted distributions. This had received numerous attentions from researchers in the field of distribution theory. Few of these works are; inverted gamma by (Abid & Al-Hassany, 2016), inverted exponential by (Abouammoh et al., 2009), inverted Weibull by (Flaih et al., 2012) to mention a few. In this paper, we proposed an inverted Dagum distribution. Some statistical properties of the proposed distribution were derived. A simulation study is further performed to check the flexibility and usefulness of the proposed distribution.

2. Proposed Distribution

Motivated by (Oluyede & Rajassooriya) and the literatures cited therein, we proposed an inverted Dagum distribution. A random variable $\boldsymbol{X}$ is said to have inverted Dagum distribution, if the following transformation is applicable $\boldsymbol{X} = \frac{1}{\boldsymbol{Y}}$. Where $\boldsymbol{Y}$ is Dagum distribution random variable with pdf and cdf expressed respectively in (1) and (2). By applying cdf technique:

$$F_Y(y) = P(Y \le y) = P\left(\frac{1}{X} \le y\right) = P\left(\frac{1}{y} \le X\right)$$

$$= P\left(X > \frac{1}{y}\right) = 1 - P\left(X \le \frac{1}{y}\right) \tag{3}$$

The pdf and cdf of the proposed distribution can be expressed as;

$$F_{IDD}(x;\theta,\alpha,\lambda) = 1 - \left[1 + \lambda x^{\alpha}\right]^{-\theta} \qquad x > 0 \tag{4}$$

$$f_{IDD}(x;\theta,\alpha,\lambda) = \theta\alpha\lambda x^{\alpha-1}\left(1 + \lambda x^{\alpha}\right)^{-\theta-1} \qquad x > 0 \tag{5}$$

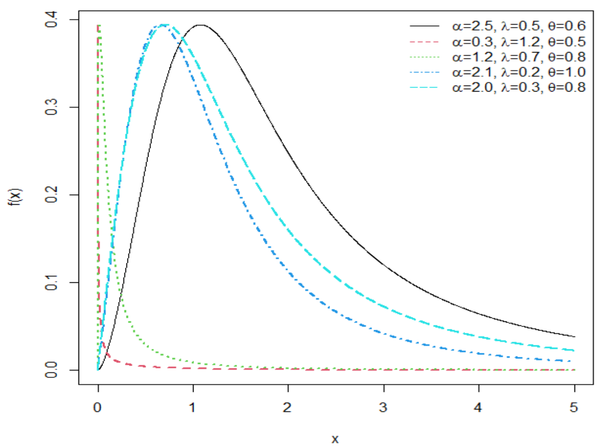

Fig.1. plot of the pdf of IDD

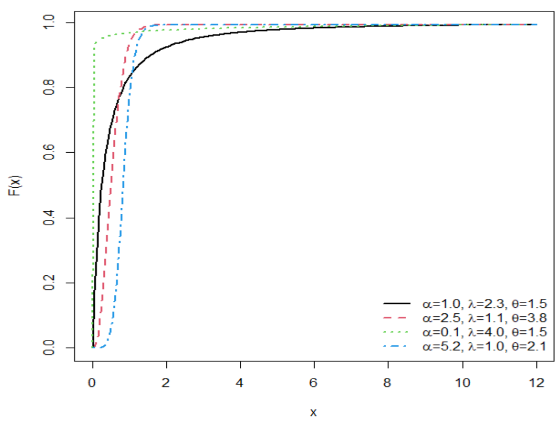

Fig.2. plot of the cdf of IDD

3. Statistical Properties

Here, some important Mathematical and Statistical properties of the proposed inverted Dagum distribution like quantile function, hazard function, moments, moment generating function, Renyi's entropy were presented.

3.1. Quantile Function

Let $\boldsymbol{X}$ be a random variable with pdf given in (5). The quantile function of $\boldsymbol{X}$ (the proposed distribution) can be expressed as;

$$Q(u;\theta,\alpha,\lambda) = \left\{\lambda^{-1}\left[(1-u)^{-\frac{1}{\theta}} - 1\right]\right\}^{\frac{1}{\alpha}} \qquad ;0 < u < 1$$

3.2. Moments

The moments of any distribution tell a lot of its features. Characteristics like tendency, skewness, dispersion, kurtosis etc can be observed through moments. If the random variable $\boldsymbol{X}$ has an inverted Dagum distribution, then its $\boldsymbol{r}^{th}$ moment about zero can be expressed as;

$$E\left(X^r\right)=\int_0^\infty x^r f(x)dx$$

$$=\theta\lambda\alpha\int_0^\infty x^{r+\alpha-1}\left(1+\lambda x^\alpha\right)^{-\theta-1}dx$$

$$=\theta\alpha\lambda^{2-r}\int_0^\infty\left(\lambda x^\alpha\right)^{r-1}\left(1+\lambda x^\alpha\right)^{-\theta-1+r-r}dx$$

$$=\theta\alpha\lambda^{2-r}\int_0^\infty\left(\lambda x^\alpha\right)^{r-1}\left(1+\lambda x^\alpha\right)^{-((r+1)+(\theta-r))}dx$$

$$=\theta\alpha\lambda^{2-r}B(1+r,\theta-r) \tag{7}$$

From (7), the mean and the variance of X can be expressed respectively as;

$$E(X)=\theta\alpha\lambda B(2,\theta-1) \tag{8}$$

$$\text{Variance}=E\left(X^2\right)-\left[E(X)\right]^2$$

$$=\theta\alpha B(3,\theta-2)-\left[\theta\alpha\lambda B(2,\theta-1)\right]^2 \tag{9}$$

3.3. Moment Generating Function

Many important features and characteristics of a distribution can be observed through its moment generating function (mgf). Let X be a random variable having an inverted Dagum distribution with pdf given as (5), using the definition of moment generating function of X utilizing (5), we have;

$$M_X(t)=E\left[e^{tx}\right]=\int_0^\infty e^{tx}f(x)dx$$

$$=\theta\lambda\alpha\int_0^\infty e^{tx}x^{\alpha-1}\left(1+\lambda x^\alpha\right)^{-\theta-1}dx$$

$$=\theta\lambda\alpha\int_0^\infty\sum_{k=0}^\infty\frac{t^k x^k}{k!}x^{\alpha-1}\left(1+\lambda x^\alpha\right)^{-\theta-1}dx$$

$$=\theta\alpha\sum_{k=0}^\infty\frac{\lambda^{2-k}t^k}{k!}\int_0^\infty\left(\lambda x^\alpha\right)^{k-1}\left(1+\lambda x^\alpha\right)^{-((r+1)+(\theta-r))}dx \tag{10}$$

After simplification (10) becomes;

$$M_X(t)=\theta\alpha M_k(\lambda)B((r+1),(\theta-r)) \tag{11}$$

Where; $M_k(\lambda)=\sum_{k=0}^\infty\frac{\lambda^{2-k}t^k}{k!}$

3.4. Survival Function

Let X be a random variable having an inverted Dagum distribution, its survival function is given by;

$$R_X(x;\theta,\alpha,\lambda)=1-F_X(x;\theta,\alpha,\lambda)$$

$$=\left[1+\lambda x^\alpha\right]^{-\theta} \tag{12}$$

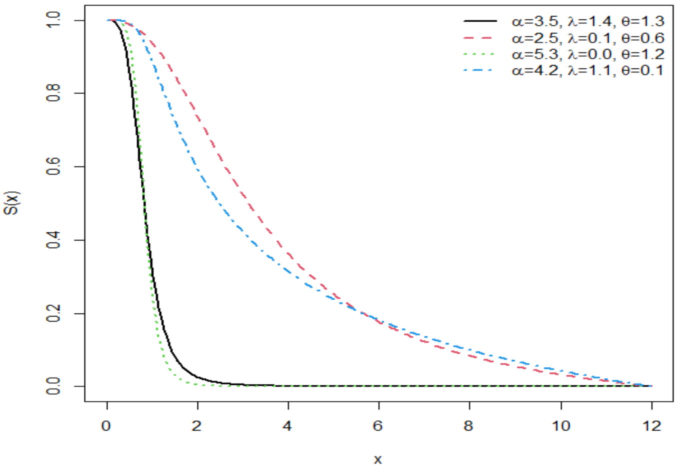

Fig.3. plot of the survival function of IDD

3.5. Hazard Function

The reliability characteristics of a system can be checked through its hazard rate function. The hazard rate function of the inverted Dagum distribution is given by;

$$\begin{aligned} h_X(x;\theta,\alpha,\lambda) &= \frac{f(x;\theta,\alpha,\lambda)}{1-F_X(x;\theta,\alpha,\lambda)} \\ &= \theta\alpha\lambda x^{\alpha-1}(1+\lambda x^{\alpha})^{-1} \end{aligned} \tag{13}$$

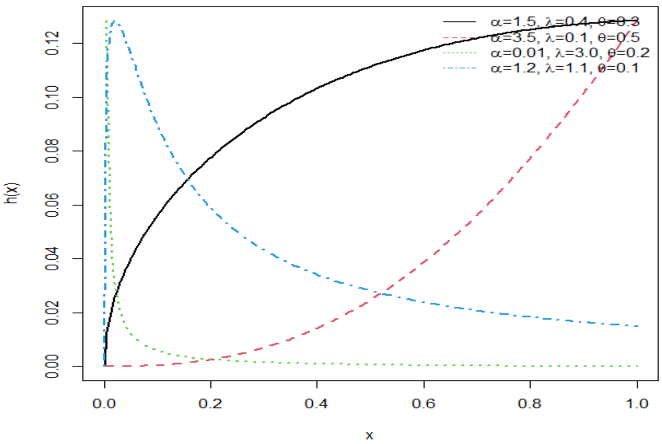

Fig.4. plot of the hazard function of IDD

3.6. Renyi Entropy

This is the measure of the variation of the uncertainty of the distribution. A large value of entropy indicates the greater uncertainty in the data. Renyi (1961) defined;

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log\left(\int_0^{\infty} f^{\gamma}(x)dx\right) \tag{14}$$

From equation (5) we have

$$f(x) = \theta\alpha\lambda x^{\alpha-1}\left(1+\lambda x^{\alpha}\right)^{-\theta-1}$$

$$f^{\gamma}(x)=\left[\theta\alpha\lambda x^{\alpha-1}\left(1+\lambda x^{\alpha}\right)^{-\theta-1}\right]^{\gamma}$$

$$f^{\gamma}(x)=(\theta\alpha)^{\gamma}\lambda^{\gamma}x^{\gamma(\alpha-1)}\left[\left(1+\lambda x^{\alpha}\right)^{-\gamma(1+\theta)}\right] \tag{15}$$

3.7. Order Statistics

Suppose $X_1, X_2, \ldots, X_n$ are random sample having an inverted Dagum distribution. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics corresponding to the samples. The pdf and cdf of the r^{th} order statistics are given as;

$$f_{r:n}(t)=\frac{\theta\alpha\lambda x^{\alpha-1}n!}{(r-1)!(n-r)!}\sum_{j=k}^{n}(-1)^{n}\binom{n-r}{u}\left(1+\lambda x_i^{\alpha}\right)^{-\theta-1}\left[1-\left(1-\lambda x^{\alpha}\right)\right]^{r-1+u} \tag{16}$$

$$F_{r:n}(x)=\sum_{j=k}^{n}\binom{n}{l}\left(1+\lambda x^{\alpha}\right)^{\theta(l-n)}\left[1-\left(1+\lambda x^{\alpha}\right)^{-\theta}\right]^{l} \tag{17}$$

3.8. Estimation

Here method of maximum likelihood method is used to estimate the parameters of the proposed inverted Dagum distribution. The likelihood function is given by;

$$L\left(\theta,\lambda,\alpha\right)=\left(\theta\lambda\alpha\right)^{n}\prod_{i=1}^{n}x_i^{\alpha-1}\left(1+\lambda x_i^{\alpha}\right)^{-\theta-1} \tag{18}$$

The log-likelihood function is;

$$\ln L\left(\theta,\lambda,\alpha\right)=n\log\theta+n\log\alpha+n\log\lambda+(\alpha-1)\sum_{i=1}^{n}\log x_i-(\theta+1)\sum_{i=1}^{n}\log\left(1+\lambda x_i^{\alpha}\right) \tag{19}$$

The log-likelihood function is maximized by differentiating (18) w.r.t to the parameters which yields;

$$\frac{\partial}{\partial\theta}\ln L(\theta,\lambda,\alpha)=\frac{n}{\theta}-\sum_{i=1}^{n}\ln\left(1+\lambda x_i^{\alpha}\right)=0 \tag{20}$$

$$\frac{\partial}{\partial\alpha}\ln L(\theta,\lambda,\alpha)=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln x_i-(\theta+1)\sum_{i=1}^{n}\frac{\alpha\lambda x_i^{\alpha-1}}{1+\lambda x_i^{\alpha}}=0 \tag{21}$$

$$\frac{\partial}{\partial\lambda}\ln L(\theta,\lambda,\alpha)=\frac{n}{\lambda}-(\theta+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}}{1+\lambda x_i^{\alpha}}=0 \tag{22}$$

The maximum likelihood estimates of the parameters can be obtained by solving the non-linear equations (20), (21), and (22) numerically.

4. Numerical Computation

Here, a numerical computation was performed to show the performance of the proposed distribution. The proposed distribution can easily be generated through the transformation $\tau=\left[\frac{1}{\lambda}\left(1-u\right)^{-\theta}-1\right]^{\frac{1}{\alpha}}$, where u is a uniform $(0,1)$. Two choices of $\theta=0.5$ and 1.5 are considered as shape parameters and the scale parameters: $\alpha=4$ and 1, $\lambda=0.1$ and 2 were used. In both cases, samples sizes of $n=50,100,250,500$ and 1000 were considered. For given $\gamma=\left(\theta,\lambda,\alpha\right)$ the means, bias and root mean square error are computed respectively as;

$$Mean=\frac{\sum\gamma_i}{n},\ Bias\left(\gamma_i\right)=\frac{1}{B}\sum_{i=1}^{B}\left(\gamma_i-\gamma\right)\text{ and }RMSE\left(\gamma\right)=\sqrt{\frac{1}{B}\sum_{i=1}^{B}\left(\gamma_i-\gamma\right)^{2}}$$

Table 1. Means, bias and RMSE of α, θ and λ for $\alpha = 4.0, \theta = 0.5$ and $\lambda = 0.1$

n	α			θ			λ		
	MEANS	BIAS	RMSE	MEANS	MEANS	BIAS	RMSE	BIAS	MEANS
20	4.5416	0.5416	1.5253	0.6184	0.1184	0.4057	0.1054	0.0054	0.1250
50	4.2997	0.2997	1.0773	0.5442	0.0442	0.2325	0.1004	0.0004	0.0549
100	4.1918	0.1918	0.7441	0.5125	0.1497	0.1497	0.0999	-0.0001	0.0375
250	4.0669	0.0669	0.4564	0.5069	0.0069	0.0942	0.0995	-0.0005	0.0224
500	4.0366	0.0366	0.3143	0.5018	0.0018	0.0664	0.1002	0.0002	0.0149
1000	4.0163	0.0163	0.2159	0.5014	0.0014	0.0462	0.1004	0.0004	0.0109

Table 2. Means, bias and RMSE of α, θ and λ for $\alpha = 1.0, \theta = 1.5$ and $\lambda = 2.0$

n	α			θ			λ		
	MEANS	BIAS	RMSE	MEANS	BIAS	RMSE	MEANS	BIAS	RMSE
20	1.0820	0.0820	0.2427	1.8485	0.3485	0.9268	2.4032	0.4032	1.8064
50	1.0343	0.0343	0.1533	1.6827	0.1827	0.6381	2.3160	0.3160	1.5318
100	1.0189	0.0189	0.1066	1.5963	0.0963	0.4689	2.1882	0.1882	1.0169
250	1.0073	0.0073	0.0746	1.5592	0.0592	0.3407	2.0970	0.0970	0.7676
500	1.0036	0.0036	0.0553	1.5287	0.0287	0.2552	2.0712	0.0712	0.5735
1000	1.0015	0.0015	0.0397	1.5168	0.0168	0.1811	2.0389	0.0389	0.4127

The simulation result is given in the Table 1 and Table 2, the performance of the maximum likelihood shows consistency because the mean estimates converge to actual parameter values and the biases and Root Mean Square Errors is decreasing as the sample size increases.

5. Conclusion

In this work we presented a new univariate continuous distribution called the inverted Dagum distribution by taking the transformation of the reciprocal of the random variable of the Dagum distribution on the ground of the type I Dagum distribution. After introducing the distribution, we obtained its basic properties like rth moments, mean, moment generating function, quantile function, hazard function, survival function, Renyi entropy and order statistics. Maximum likelihood method was used to provide the estimates of the model parameters. We also provided the density hazard rate plots of the distribution with some assumed values. Additionally, a simulation study was performed to show the performance of the proposed distribution.

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